Error-correcting Pairs for a Public-key Cryptosystem

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Code-based Cryptography Workshop 2012
Lyngby, 9 May 2012

Introduction and content

- Error-correcting pair
  - Generalized Reed-Solomon codes
  - Alternant codes
  - Goppa codes
- t-error-correcting pair corrects t-errors
- Algebraic geometry codes
- Code-based cryptography

C linear block code: $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$

parameters $[n, k, d]$:
n = length
k = dimension of C
d = minimum distance of C

$$d = \min |\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}|$$

t = error-correcting capacity of C

$$t = \lfloor \frac{d(C) - 1}{2} \rfloor$$

Inner and star product

The standard inner product is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

For two subsets A and B of $\mathbb{F}_q^n$
$A \perp B$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$ for all $\mathbf{a} \in A$ and $\mathbf{b} \in B$

Let $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{F}_q^n$
The star product is defined by coordinatewise multiplication:

$$\mathbf{a} * \mathbf{b} = (a_1 b_1, \ldots, a_n b_n)$$

For two subsets A and B of $\mathbb{F}_q^n$

$$A * B = \{\mathbf{a} * \mathbf{b} \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$$

Let C be a linear code in $\mathbb{F}_q^n$

The pair (A, B) of linear subcodes of $\mathbb{F}_{q^m}^n$ is a called a t-error correcting pair (ECP) over $\mathbb{F}_{q^m}$ for C if

- E.1 $(A * B) \perp C$
- E.2 $k(A) > t$
- E.3 $d(B^\perp) > t$
- E.4 $d(A) + d(C) > n$

Generalized Reed-Solomon codes

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of mutually distinct elements of $\mathbb{F}_q$
Let $\mathbf{b} = (b_1, \ldots, b_n)$ be an n-tuple of nonzero elements of $\mathbb{F}_q$
Evaluation map:

$$\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) = (f(a_1)b_1, \ldots, f(a_n)b_n)$$

$$GRS_k(\mathbf{a}, \mathbf{b}) = \{ \, \mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) \mid f(X) \in \mathbb{F}_q[X], \deg(f(X) < k \, \}$$

Parameters: $[n, k, n-k+1]$ if $k \leq n$
Furthermore

$$\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) * \mathrm{ev}_{\mathbf{a},\mathbf{c}}(g(X)) = \mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)g(X)) * \mathbf{c}$$

$$\langle GRS_k(\mathbf{a}, \mathbf{b}) * GRS_l(\mathbf{a}, \mathbf{c}) \rangle = GRS_{k+l-1}(\mathbf{a}, \mathbf{b} * \mathbf{c})$$

t-ECP for $GRS_{n-2t}(\mathbf{a}, \mathbf{b})$

Let $C = GRS_{n-2t}(\mathbf{a}, \mathbf{b})$
Then C has parameters: $[n, n-2t, 2t+1]$
and $C^\perp = GRS_{2t}(\mathbf{a}, \mathbf{c})$ for some $\mathbf{c}$

Let $A = GRS_{t+1}(\mathbf{a}, \mathbf{1})$ and $B = GRS_t(\mathbf{a}, \mathbf{c})$
Then $A * B \subseteq C^\perp$

A has parameters $[n, t+1, n-t]$
B has parameters $[n, t, n-t+1]$
So $B^\perp$ has parameters $[n, n-t, t+1]$

Hence (A, B) is a t-error-correcting pair for C

Conversely an $[n, n-2t, 2t+1]$ code that has a t-ECP
is a GRS code

Alternant codes

Let **a** be an n-tuple of mutually distinct elements of $\mathbb{F}_{q^m}$
Let **b** be an n-tuple of nonzero elements of $\mathbb{F}_{q^m}$

Let $GRS_k(\mathbf{a}, \mathbf{b})$ be the GRS code over $\mathbb{F}_{q^m}$ of dimension k

The alternant code $ALT_r(\mathbf{a}, \mathbf{b})$ is the $\mathbb{F}_q$-linear restriction

$$ALT_r(\mathbf{a}, \mathbf{b}) = \mathbb{F}_q^n \cap (GRS_r(\mathbf{a}, \mathbf{b}))^{\perp}$$

Then $ALT_r(\mathbf{a}, \mathbf{b})$ has parameters $[n, k, d]_q$ with

$$k \geq n - mr \quad \text{and} \quad d \geq r + 1$$

Every linear code of minimum distance at least 2 is an alternant code!

t-ECP for $ALT_{2t}(\mathbf{a}, \mathbf{b})$

Let $C = ALT_{2t}(\mathbf{a}, \mathbf{b})$
Then C has minimum distance $d \geq 2t + 1$
and $C \subseteq (GRS_{2t+1}(\mathbf{a}, \mathbf{b}))^{\perp}$

Let $A = GRS_{t+1}(\mathbf{a}, \mathbf{1})$ and $B = GRS_{t}(\mathbf{a}, \mathbf{b})$
Then $A * B \subseteq GRS_{2t+1}(\mathbf{a}, \mathbf{b})$
Then $(A * B) \perp C$

A has parameters $[n, t + 1, n - t]$
B has parameters $[n, t, n - t + 1]$
So $B^{\perp}$ has parameters $[n, n - t, t + 1]$

Hence (A, B) is a t-error-correcting pair over $\mathbb{F}_{q^m}$ for C

Goppa codes

Let $L = (a_1, \ldots, a_n)$ be an n-tuple of n distinct elements of $\mathbb{F}_{q^m}$
Let g be a polynomial with coefficients in $\mathbb{F}_{q^m}$ such that

$$g(a_j) \neq 0 \text{ for all } j$$

Then g is called Goppa polynomial with respect to L

Define the $\mathbb{F}_q$-linear Goppa code $\Gamma(L, g)$ by

$$\Gamma(L, g) = \left\{ \mathbf{c} \in \mathbb{F}_q^n \mid \sum_{j=1}^{n} \frac{c_j}{X - a_j} \equiv 0 \bmod g(X) \right\}$$

Goppa codes are alternant codes

Let $L = \mathbf{a} = (a_1, \ldots, a_n)$
Let g be a Goppa polynomial of degree r

Let $b_j = 1/g(a_j)$
Then

$$\Gamma(L, g) = ALT_r(\mathbf{a}, \mathbf{b})$$

Hence $\Gamma(L, g)$ has parameters $[n, k, d]_q$ with

$$k \geq n - mr \text{ and } d \geq r + 1$$

and has an $\lfloor r/2 \rfloor$-error-correcting pair

Binary Goppa codes

Let $L = \mathbf{a} = (a_1, \ldots, a_n)$
Let g be a Goppa polynomial with coefficients in $\mathbb{F}_{2^m}$ of degree r

Suppose moreover that g has no square factor
Then

$$\Gamma(L, g) = \Gamma(L, g^2)$$

Hence $\Gamma(L, g)$ has parameters $[n, k, d]_q$ with

$$k \geq n - mr \text{ and } d \geq 2r + 1$$

and has an r-error-correcting pair

Let C be a linear code in $\mathbb{F}_q^n$

The pair (A, B) of linear subcodes of $\mathbb{F}_{q^m}^n$ is a called a t-error correcting pair (ECP) over $\mathbb{F}_{q^m}$ for C if

E.1 $(A * B) \perp C$

E.2 $k(A) > t$

E.3 $d(B^\perp) > t$

E.4 $d(A) + d(C) > n$

Let (A, B) be linear subcodes of $\mathbb{F}_{q^m}^n$ that satisfy $E.1$, $E.2$, $E.3$ and

E.5 $d(A^\perp) > 1$

E.6 $d(A) + 2t > n$

Then $d(C) \geq 2t + 1$ and (A, B) is a t-ECP for C

Kernel of a received word

Let A and B be linear subspaces of $\mathbb{F}_{q^m}^n$
Let $\mathbf{r} \in \mathbb{F}_q^n$ be a received word
Define the kernel

$$K(\mathbf{r}) = \{ \mathbf{a} \in A \mid (\mathbf{a} * \mathbf{b}) \cdot \mathbf{r} = 0 \text{ for all } \mathbf{b} \in B \}$$

Lemma
Let C be an $\mathbb{F}_q$-linear code of length n
Let $\mathbf{r}$ be a received word with error vector $\mathbf{e}$
So $\mathbf{r} = \mathbf{c} + \mathbf{e}$ for some $\mathbf{c} \in C$
If $A * B \subseteq C^\perp$, then

$$K(\mathbf{r}) = K(\mathbf{e})$$

Kernel for a GRS code

Let $A = GRS_{t+1}(\mathbf{a}, \mathbf{1})$ and $B = GRS_t(\mathbf{a}, \mathbf{1})$ and $C = \langle A * B \rangle^\perp$

Let
$\mathbf{a}_i = \mathrm{ev}_{\mathbf{a},\mathbf{1}}(X^{i-1})$ for $i = 1, \ldots, t+1$
$\mathbf{b}_j = \mathrm{ev}_{\mathbf{a},\mathbf{1}}(X^{j})$ for $j = 1, \ldots, t$
$\mathbf{h}_l = \mathrm{ev}_{\mathbf{a},\mathbf{1}}(X^{l})$ for $l = 1, \ldots, 2t$

Then
$\mathbf{a}_1, \ldots, \mathbf{a}_{t+1}$ is a basis of A
$\mathbf{b}_1, \ldots, \mathbf{b}_t$ is a basis of B
$\mathbf{h}_1, \ldots, \mathbf{h}_{2t}$ is a basis of $C^\perp$

Furthermore

$$\mathbf{a}_i * \mathbf{b}_j = \mathrm{ev}_{\mathbf{a},\mathbf{1}}(X^{i+j-1}) = \mathbf{h}_{i+j-1}$$

Matrix of syndromes for a GRS code

Let r be a received word and
$\mathbf{s} = \mathbf{r}H^T$ its syndrome
Then

$$(\mathbf{b}_j * \mathbf{a}_i) \cdot \mathbf{r} = s_{i+j-1}.$$

To compute the kernel $K(\mathbf{r})$ we have to compute the null space of the matrix of syndromes

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_t & s_{t+1} \\ s_2 & s_3 & \cdots & s_{t+1} & s_{t+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_t & s_{t+1} & \cdots & s_{2t-1} & s_{2t} \end{pmatrix}$$

Let (A, B) be a t-ECP for C
Let J be a subset of $\{1, \ldots, n\}$
Define the subspace of A

$$A(J) = \{\, \mathbf{a} \in A \mid a_j = 0 \ \text{ for all } \ j \in J \,\}$$

Lemma

Let $(A * B) \perp C$
Let $\mathbf{e}$ be an error vector of the received word $\mathbf{r}$
If $I = \mathrm{supp}(\mathbf{e}) = \{\, i \mid e_i \neq 0 \,\}$, then

$$A(I) \subseteq K(\mathbf{r})$$

If moreover $d(B^{\perp}) > \mathrm{wt}(\mathbf{e})$, then $A(I) = K(\mathbf{r})$

Basic algorithm

Let (A, B) be a t-ECP for C with $d(C) \geq 2t + 1$
Suppose that $\mathbf{c} \in C$ is the code word sent and $\mathbf{r} = \mathbf{c} + \mathbf{e}$ is the received word for some error vector $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) \leq t$

The basic algorithm for the code C:

- Compute the kernel $K(\mathbf{r})$

  This kernel is nonzero since $k(A) > t$

- Take a nonzero element $\mathbf{a}$ of $K(\mathbf{r})$

  $K(\mathbf{r}) = K(\mathbf{e})$ since $(A * B) \perp C$

- Determine the set J of zero positions of $\mathbf{a}$

  $\mathrm{supp}(\mathbf{e}) \subseteq J$ since $d(B^{\perp}) > t$

  $|J| < d(C)$ since $d(A) + d(C) < n$

- Compute the error values by erasure decoding

Theorem

Let C be an $\mathbb{F}_q$-linear code of length n
Let (A, B) be a t-error-correcting pair over $\mathbb{F}_{q^m}$ for C

Then the basic algorithm corrects t errors
for the code C with complexity $\mathcal{O}((mn)^3)$

Algebraic geometry codes

Let $\mathcal{X}$ be an algebraic variety over $\mathbb{F}_q$
with a subset $\mathcal{P}$ of $\mathcal{X}(\mathbb{F}_q)$ enumerated by $P_1, \ldots, P_n$

Suppose that we have a vector space L over $\mathbb{F}_q$
of functions on $\mathcal{X}$ with values in $\mathbb{F}_q$
So $f(P_i) \in \mathbb{F}_q$ for all i and $f \in L$
In this way we have an evaluation map

$$ev_{\mathcal{P}} : L \longrightarrow \mathbb{F}_q^n$$

defined by $ev_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n))$

This evaluation map is linear, so its image is a linear code

Codes on the affine line

The classical example:
Generalized Reed-Solomon codes

The geometric object $\mathcal{X}$ is the affine line over $\mathbb{F}_q$
The points are n distinct elements of $\mathbb{F}_q$
L is the vector space of polynomials of degree at most $k-1$
and with coefficients in $\mathbb{F}_q$

This vector space has dimension k
Such polynomials have at most $k-1$ zeros
so nonzero codewords have at least $n-k+1$ nonzeros

This code has parameters $[n, k, n-k+1]$ if $k \leq n$

Codes on curves-function fields

Let $\mathcal{X}$ be an algebraic curve over $\mathbb{F}_q$ of genus g
$\mathbb{F}_q(\mathcal{X})$ is the function field of the curve $\mathcal{X}$ with field of constants $\mathbb{F}_q$

Let f be a nonzero rational function on the curve
The divisor of zeros and poles of f is denoted by (f)

Let E be a divisor of $\mathcal{X}$ of degree m
Then

$$L(E) = \{ f \in \mathbb{F}_q(\mathcal{X}) \mid f = 0 \text{ or } (f) \geq -E \}$$

The dimension of the space $L(E)$ is denoted by $l(E)$
Then $l(E) \geq m + 1 - g$ and equality holds if $m > 2g - 2$
by the Theorem of Riemann-Roch

Let $\mathcal{P} = (P_1, \ldots, P_n)$ an n-tuple of mutual distinct points of $\mathcal{X}(\mathbb{F}_q)$

If the support of E is disjoint from $\mathcal{P}$, then the evaluation map

$$\mathrm{ev}_{\mathcal{P}} : L(E) \to \mathbb{F}_q^n$$

where $\mathrm{ev}_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n))$, is well defined.

The algebraic geometry code $C_L(\mathcal{X}, \mathcal{P}, E)$
is the image of $L(E)$ under the evaluation map $\mathrm{ev}_{\mathcal{P}}$
If $m < n$, then $C_L(\mathcal{X}, \mathcal{P}, E)$ is an $[n, k, d]$ code with

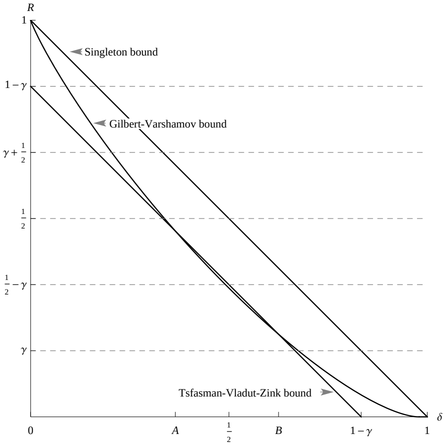
$$k \geq m + 1 - g \text{ and } d \geq n - m$$

$n - m$ is called the designed minimum distance of $C_L(\mathcal{X}, \mathcal{P}, E)$

Information rate

Information rate	$R = k/n$
Relative minimum distance	$\delta = d/n$
Singleton	$R + \delta \leq 1$
Gilbert-Varshamov	$R \geq 1 - H_q(\delta)$
q-ary entropy function	H_q
Goppa for AG codes	$R + \delta \geq 1 - \gamma$
Relative genus	$\gamma = g/n$
Ihara-Tsfasman-Vladut-Zink	$\gamma = \frac{1}{\sqrt{q}-1}$

Bounds on codes



Figuur: Bounds on R as a function of δ for $q = 49$ and $\gamma = \frac{1}{6}$.

Dual codes on curves

Let ω be a differential form with
a simple pole at P_j with residue 1 for all $j = 1, \ldots, n$

Let K be the canonical divisor of ω
Let m be the degree of the divisor E on $\mathcal{X}$
with disjoint support from $\mathcal{P}$

Let $E^{\perp} = D - E + K$ and $m^{\perp} = \deg(E^{\perp})$
Then $m^{\perp} = 2g - 2 - m + n$ and

$$C_L(\mathcal{X}, \mathcal{P}, E)^{\perp} = C_L(\mathcal{X}, \mathcal{P}, E^{\perp})$$

$m - 2g + 2$ is called the designed minimum distance of $C_L(\mathcal{X}, \mathcal{P}, E)^{\perp}$

Let F and G be divisors
Then there is a well defined linear map

$$L(F) \otimes L(G) \longrightarrow L(F+G)$$

given on generators by

$$f \otimes g \mapsto fg$$

Hence

$$C_L(\mathcal{X}, \mathcal{P}, F) * C_L(\mathcal{X}, \mathcal{P}, G) \subseteq C_L(\mathcal{X}, \mathcal{P}, F+G)$$

ECP for AG codes - 2

Let $C = C_L(\mathcal{X}, \mathcal{P}, E)^\perp$

Choose a divisor F with support disjoint from $\mathcal{P}$
Let $A = C_L(\mathcal{X}, \mathcal{P}, F)$
Let $B = C_L(\mathcal{X}, \mathcal{P}, E - F)$

Then
- $A * B \subseteq C^\perp$
- If $t + g \leq \deg(F) < n$, then $k(A) > t$
- If $\deg(G - F) > t + 2g - 2$, then $d(B^\perp) > t$
- If $\deg(G - F) > 2g - 2$, then $d(A) + d(C) > n$

Proposition

An algebraic geometry code of
designed minimum distance d
from a curve over $\mathbb{F}_q$ of genus g
has a t-error-correcting pair over $\mathbb{F}_q$ where

$$t = \lfloor \frac{d-1-g}{2} \rfloor$$

ECP for AG codes - improvement

Proposition

An algebraic geometry code of
designed minimum distance d
from a curve over $\mathbb{F}_q$ of genus g
has a t-error-correcting pair over $\mathbb{F}_{q^m}$ where

$$t = \lfloor \frac{d-1}{2} \rfloor$$

if

$$m > \log_q \left(2\binom{n}{t} + 2\binom{n}{t+1} + 1 \right)$$

By randomnization - Not constructive!

Public-key cryptosystems - 1

Koblitz:

At the heart of any public-key cryptosystem is a one-way function - a function

$$y = f(x)$$

that is easy to evaluate but for which is computationally infeasible (one hopes) to find the inverse

$$x = f^{-1}(y)$$

Public-key cryptosystems - 2

PKC systems use trapdoor one-way functions

by mathematical problems that are (supposedly) hard

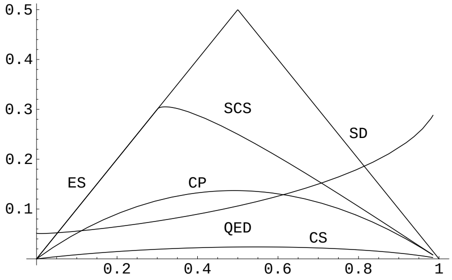
RSA, factoring integers: given $n = pq$ find (p, q)
Diffie-Hellman, discrete-log problem in $\mathbb{F}_q$: given $b = a^n$ find n
Elliptic curve PKC, addition on elliptic curve: given $Q = nP$, find n

Code based PKC systems, decoding of codes

McEliece (Goppa codes)
Niederreiter with parity check matrix instead of generator matrix
Janwa-Moreno (Algebraic geometry codes)

Decoding arbitrary linear codes

Exponential complexity $\approx q^{e(R)n}$



x-axis: information rate $R = k/n$

y-axis: complexity exponent $e(R)$

McEliece:
Let $\mathcal{C}$ be a class of codes that have
efficient decoding algorithms correcting t errors with $t \leq (d-1)/2$

Secret key: (S, G, P)
S an invertible $k \times k$ matrix
G a $k \times n$ generator matrix of a code C in $\mathcal{C}$.
P an $n \times n$ permutation matrix

Public key: $G' = SGP$

Message: $\mathbf{m}$ in $\mathbb{F}_q^k$
Encryption: $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$ with random chosen $\mathbf{e}$ in $\mathbb{F}_q^n$ of weight t
Decryption: $\mathbf{y}P^{-1} = \mathbf{m}SG + \mathbf{e}P^{-1}$ and $\mathbf{e}P^{-1}$ has weight t
Decoder gives $\mathbf{c} = \mathbf{m}SG$ as closest codeword

Code based PKC systems - 2

G, S and P are kept secret
$G' = SGP$ is public

The (trapdoor) one-way function of
the McEliece public cryptosystem is given by

$$x = (\mathbf{m}, \mathbf{e}) \mapsto y = \mathbf{m}G' + \mathbf{e}$$

where $\mathbf{m} \in \mathbb{F}_q^k$ is the plaintext
$\mathbf{e} \in \mathbb{F}_q^n$ is a random error vector with hamming weight at most t

Let $\mathcal{C}_{ECP}$ be the set of pairs (A, B) that satisfy $E.2$, $E.3$, $E.5$ and $E.6$

The McEliece cryptosystem on codes $C \subseteq (A * B)^{\perp}$ with (A, B) in $\mathcal{C}_{ECP}$ is based on the inherent tractability of finding an inverse on the one-way function

$$x = (A, B) \mapsto y = (A * B)$$

where (A, B) is in $\mathcal{C}_{ECP}$

Code based PKC systems - 4

State of the art

- GRS codes: solved by Sidelnikov-Shestakov
- Alternant codes: open
- Goppa codes: open
- AG codeds: work in progress by

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  Edgar Martínez-Moro
  Ruud Pellikaan
  Diego Ruano